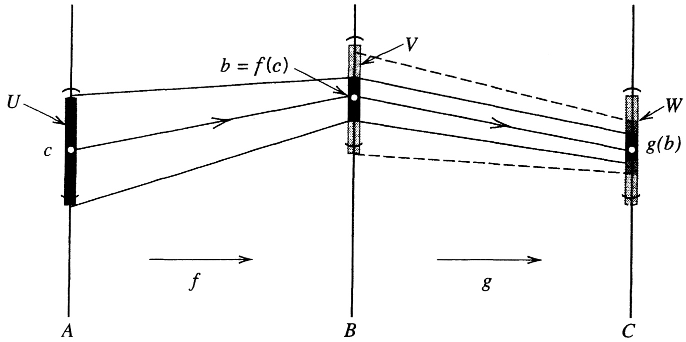

Figure 5.2.1 The composition of f and g

Theorems 5.2.6 and 5.2.7 are very useful in establishing that certain functions are continuous. They can be used in many situations where it would be difficult to apply the definition of continuity directly.

5.2.8 Examples **(a)** Let $g_1(x) := |x|$ for $x \in \mathbb{R}$. It follows from the Triangle Inequality that

$$|g_1(x) - g_1(c)| \leq |x - c|$$

for all $x, c \in \mathbb{R}$. Hence g_1 is continuous at $c \in \mathbb{R}$. If $f : A \to \mathbb{R}$ is any function that is continuous on A, then Theorem 5.2.7 implies that $g_1 \circ f = |f|$ is continuous on A. This gives another proof of Theorem 5.2.4.

(b) Let $g_2(x) := \sqrt{x}$ for $x \geq 0$. It follows from Theorems 3.2.10 and 5.1.3 that g_2 is continuous at any number $c \geq 0$. If $f : A \to \mathbb{R}$ is continuous on A and if $f(x) \geq 0$ for all $x \in A$, then it follows from Theorem 5.2.7 that $g_2 \circ f = \sqrt{f}$ is continuous on A. This gives another proof of Theorem 5.2.5.

(c) Let $g_3(x) := \sin x$ for $x \in \mathbb{R}$. We have seen in Example 5.2.3(c) that g_3 is continuous on $\mathbb{R}$. If $f : A \to \mathbb{R}$ is continuous on A, then it follows from Theorem 5.2.7 that $g_3 \circ f$ is continuous on A.

In particular, if $f(x) := 1/x$ for $x \neq 0$, then the function $g(x) := \sin(1/x)$ is continuous at every point $c \neq 0$. [We have seen, in Example 5.1.8(a), that g cannot be defined at 0 in order to become continuous at that point.] □

Exercises for Section 5.2

1. Determine the points of continuity of the following functions and state which theorems are used in each case.

   (a) $f(x) := \dfrac{x^2 + 2x + 1}{x^2 + 1} \quad (x \in \mathbb{R})$, (b) $g(x) := \sqrt{x + \sqrt{x}} \quad (x \geq 0)$,

   (c) $h(x) := \dfrac{\sqrt{1 + |\sin x|}}{x} \quad (x \neq 0)$, (d) $k(x) := \cos\sqrt{1 + x^2} \quad (x \in \mathbb{R})$.

2. Show that if $f : A \to \mathbb{R}$ is continuous on $A \subseteq \mathbb{R}$ and if $n \in \mathbb{N}$, then the function f^n defined by $f^n(x) = (f(x))^n$, for $x \in A$, is continuous on A.

3. Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that (a) the sum $f + g$ is continuous at c, (b) the product fg is continuous at c.

4. Let $x \mapsto [\![x]\!]$ denote the greatest integer function (see Exercise 5.1.4). Determine the points of continuity of the function $f(x) := x - [\![x]\!]$, $x \in \mathbb{R}$.
5. Let g be defined on $\mathbb{R}$ by $g(1) := 0$, and $g(x) := 2$ if $x \neq 1$, and let $f(x) := x + 1$ for all $x \in \mathbb{R}$. Show that $\lim\limits_{x \to 0} g \circ f \neq (g \circ f)(0)$. Why doesn't this contradict Theorem 5.2.6?
6. Let f, g be defined on $\mathbb{R}$ and let $c \in \mathbb{R}$. Suppose that $\lim\limits_{x \to c} f = b$ and that g is continuous at b. Show that $\lim\limits_{x \to c} g \circ f = g(b)$. (Compare this result with Theorem 5.2.7 and the preceding exercise.)
7. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.
8. Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers r. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?
9. Let $h : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ satisfying $h(m/2^n) = 0$ for all $m \in \mathbb{Z}, n \in \mathbb{N}$. Show that $h(x) = 0$ for all $x \in \mathbb{R}$.
10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$, and let $P := \{x \in \mathbb{R} : f(x) > 0\}$. If $c \in P$, show that there exists a neighborhood $V_\delta(c) \subseteq P$.
11. If f and g are continuous on $\mathbb{R}$, let $S := \{x \in \mathbb{R} : f(x) \geq g(x)\}$. If $(s_n) \subseteq S$ and $\lim(s_n) = s$, show that $s \in S$.
12. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **additive** if $f(x + y) = f(x) + f(y)$ for all x, y in $\mathbb{R}$. Prove that if f is continuous at some point x_0, then it is continuous at every point of $\mathbb{R}$. (See Exercise 4.2.12.)
13. Suppose that f is a continuous additive function on $\mathbb{R}$. If $c := f(1)$, show that we have $f(x) = cx$ for all $x \in \mathbb{R}$. [*Hint*: First show that if r is a rational number, then $f(r) = cr$.]
14. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy the relation $g(x + y) = g(x)\,g(y)$ for all x, y in $\mathbb{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point of $\mathbb{R}$. Also if we have $g(a) = 0$ for some $a \in \mathbb{R}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.
15. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at a point c, and let $h(x) := \sup\{f(x), g(x)\}$ for $x \in \mathbb{R}$. Show that $h(x) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$ for all $x \in \mathbb{R}$. Use this to show that h is continuous at c.

Section 5.3 Continuous Functions on Intervals

Functions that are continuous on intervals have a number of very important properties that are not possessed by general continuous functions. In this section, we will establish some deep results that are of considerable importance and that will be applied later. Alternative proofs of these results will be given in Section 5.5.

5.3.1 Definition A function $f : A \to \mathbb{R}$ is said to be **bounded on** A if there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$.

In other words, a function is bounded on a set if its range is a bounded set in $\mathbb{R}$. To say that a function is *not* bounded on a given set is to say that no particular number can serve as a bound for its range. In exact language, a function f is not bounded on the set A if given any $M > 0$, there exists a point $x_M \in A$ such that $|f(x_M)| > M$. We often say that f is **unbounded on** A in this case.

For example, the function f defined on the interval $A := (0, \infty)$ by $f(x) := 1/x$ is not bounded on A because for any $M > 0$ we can take the point $x_M := 1/(M + 1)$ in A to get $f(x_M) = 1/x_M = M + 1 > M$. This example shows that continuous functions need not be